To Pursue or to Evade—That is the Question

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The one-on-one air combat problem has been analyzed under a variety of assumptions regarding the aircraft dynamics and the weapon-system characteristics. However, most of these studies have not considered the problem of role determination and the possibility of real-time implementation of the derived guidance laws. These questions are addressed for a simple but plausible dynamic model of the problem. The two capture regions, the mutual kill and draw regions, are found for a single value of weapon range. In addition, the min-max optimal time controls for both are found when either is in the capture region of the other. Finally, the feasibility of applying these guidance laws is discussed.

Introduction

MANY published papers have dealt with the differential-game representation of the one-on-one air combat problem,[1-7] but relatively few have considered the question of role determination.[8-10] Since realistic representations of the problem require that *either* aircraft, in principle, can win from *some* relative geometries, it is of basic importance to find the optimal pursuit-evasion maneuvers associated with the capture regions or "sure-kill" geometries for *both* aircraft.

The aim of this study is to apply differential-game theory to a "game of two cars" dynamic model[11] of the air-combat problem. The procedure used here treats the role-specification and maneuver-determination problems, in that order, assuming that the opposing aircraft and their weapons are not significantly different. In particular, the combatants are specialized as *identical* in all respects, and each knows everything about the other. This means that geometric symmetries in the third-order state can be exploited. The hope is that actual application of the method will be feasible, since the simple dynamic model requires a minimum of data regarding state and parameters. Clearly, we are not aiming for fidelity in the model, but, instead, are hoping to find a guidance law that can be implemented easily and retains the essence of the question, "Pursue or evade?"

For an arbitrary geometry, the range vector and the two actual velocities define two separate planes in space, as in Fig. 1a. The range and *relative* velocity vectors define a single plane, however, with the two normal components of actual velocity being equal. In the analysis and results to follow, these normal velocities are ignored, since they have no kinematic effect on the optimal turn-rate controls. Of course, test simulation must include these normal velocity effects, but the analysis is done in the coplanar system of Fig. 1b.

As shown in this figure, the position and heading of aircraft B relative to aircraft A are indicated by (x,y,H) in axes fixed to A. The kill zones or terminal surfaces[11] are line segments in the direction of the velocities of each, and the final heading is unconstrained.[10] The forward-firing "envelope" is meant to represent either guns or missiles, with only the range as a parameter. Generalizing the shape of this envelope obviously complicates matters, so this generalization is left as an exercise for the reader.

The velocities and limit turn rates of A and B are assumed equal and constant. This is a troubling assumption for most, since combat aircraft are never identical; e.g., one F-15 will have an edge in top speed over another F-15, even at the same flight condition. Furthermore, aircraft speeds can vary over a wide range, and hard turns generate large drag forces, which can cause speed changes as well. But, again, the results of this study are meant to apply to aircraft which are dynamically similar in a "more or less" context. Since combat aircraft are usually "equal" within 20-40% or so, the identical features assumed should at least provide a reference solution to which later refinements can be added.

The purpose of all of these simplfications is to transform a problem we cannot solve into a simpler one which we *can* solve. If the results can be actually implemented as a guidance law, the significance of the approximations can be tested in piloted one-on-one simulations. This is the practical measure of value of the results.

Many would reasonably argue that the proposed dynamic model and kill zone are far too simplified for any practical use to come from the analysis. On the other hand, any unconstrained version of the one-on-one air combat problem has the following characteristics which help justify the model.

1) The relative position and velocity vectors define a single plane in space for any initial condition. This plane is undefined only if the geometry is colinear, i.e., head-on or tail-chase.

2) Neglecting pilot and attitude dynamics, the optimal normal accelerations of both A and B are applied in this plane. In general, this plane is tilted with respect to the horizontal during an engagement.

3) Air-to-air weapon ranges are so much greater than aircraft dimensions that point-mass, low-order dynamics are appropriate for describing the relative motion. This representation is particularly apt just before termination or capture of one by the other, and this short-term sort of maneuver is the kind which results when the speeds are equal.

Another motivation is given by earlier studies of the simpler ship collision-avoidance problem.[12] Here it was found that optimal (maximum miss-distance) maneuvers are surprisingly insensitive to realistic variations in the speed and maximum turn-rate parameters. That is, starboard- and port-turn regions for a range of values of speeds and turn rates were very similar from an operational standpoint, so accurate estimates of these parameters appeared unnecessary. The maneuvers and miss distance at short range, however, often depend strongly on the state (x,y,H). The range, bearing, and relative heading, therefore, are the important independent variables in the control algorithm.

Differential games or "min-max" control problems are found to be more difficult to solve than equivalent "max-

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max" optimization problems. It is common to find extreme complexity[9,10] in the solutions to apparently simple models of the combat problem. Therefore, if a workable pursuit-evasion guidance law cannot be developed under the stated simplifying assumptions, role and maneuver determination by the methods of differential games will have to remain a fascinating problem of only academic interest, as far as air combat is concerned. The following analysis and results should answer this question regarding practical usefulness.

Role Determination and Necessary Conditions

Both aircraft are given unit speed and unit maximum turn rate in the plane of the relative motion; this fixes the units of length and time. The common kill range is the normalized range β, and the evader must be dead ahead of the pursuer at termination. This constraint allows both aircraft to be pursuers, both being aware of the identical dynamics and weapon of the other in this differential game of full information. The roles can then be specified using the following geometric observation: When the roles and turn-rate controls are known at the relative position and heading (x,y,H), the roles and controls are *both* reversed at (x',y',H'), where

$$x' = -x\cos H + y\sin H$$
$$y' = -x\sin H - y\cos H$$
$$H' = -H \qquad (1)$$

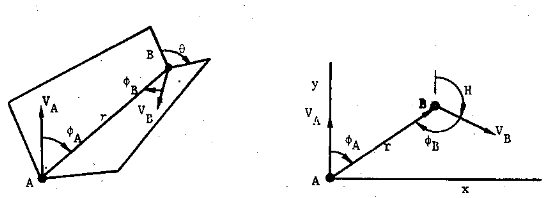
These equations merely transform the state of aircraft B relative to A to A's state relative to B in the plane of the relative position and relative velocity. This static symmetry implies that the role-reversal or mutual-kill conditions are

$$H = 180\ \text{deg} \qquad (2)$$

and

$$R(x,y,H) = x\sin(H/2) + y\cos(H/2) = 0 \qquad (3)$$

as shown in Fig. 2. These are angular heading symmetries about the relative range, and both serve as stop conditions in the backward integration of the relative motion equations from an arbitrary end condition. More generally, when speeds and turn rates differ, the equivalent role-reversal loci are initial geometries which lead to simultaneous kill in the head-on or collision geometry when both pursue. The states and trajectories for which A pursues B require $R>0$, and conversely. These necessary conditions specify roles by static considerations alone, both aircraft being assumed to have a weapon of infinite range, and each aware of the other's. Whether the pursuer actually can *succeed* in capturing the evader depends additionally on the common weapon range, β. This parameter is used with the dynamic equations of relative motion to determine controls and capture regions for the *specified* roles of the two aircraft. In addition, the mutual kill or Kamikaze end condition, represented by the intersection of the surfaces of Eqs. (2) and (3), at the range β, can be an option of the evader for certain geometries. These relative positions and headings will also be determined.



a) Arbitrary. b) Coplanar.

Fig. 1 Relative geometry.

The equations of relative motion describe the changes in position and heading of aircraft B relative to A; in Cartesian coordinates (x,y,H), these are

$$\dot{x} = -\omega_A y + \sin H$$
$$\dot{y} = -1 + \omega_A x + \cos H$$
$$\dot{H} = -\omega_A + \omega_B \qquad (|\omega_i| \leq 1, \quad i = \text{A,B}) \qquad (4)$$

In coplanar range-angle off coordinates $(r,\phi_A\phi_B)$, the dynamic equations are more symmetric:

$$\dot{r} = -\cos\phi_A - \cos\phi_B$$
$$\dot{\phi}_A = -\omega_A + (\sin\phi_A - \sin\phi_B)/r$$
$$\dot{\phi}_B = -\omega_B + (\sin\phi_A - \sin\phi_B)/r \qquad (5)$$

The kill zone is the line segment of length β. In physical units, the minimum-turn radius is V^2/ng, where n is the maximum load factor, and the weapon range is β times this radius. There is no constraint on the final relative heading H_f, but the evader must be dead ahead of the pursuer at temination. As with the terminal angle-off, including limits on the final heading[10] is in the class of "suggestions for further work."

Assuming that A is the pursuer, the necessary conditions on the controls ω_A, ω_B in the "game of kind"[11] are expressed in terms of the adjoint vector (V_x, V_y, V_H) and the state

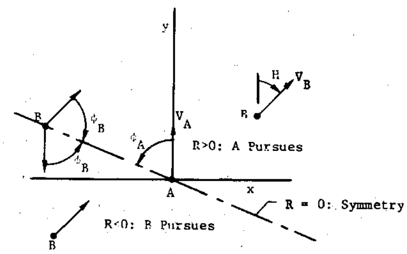
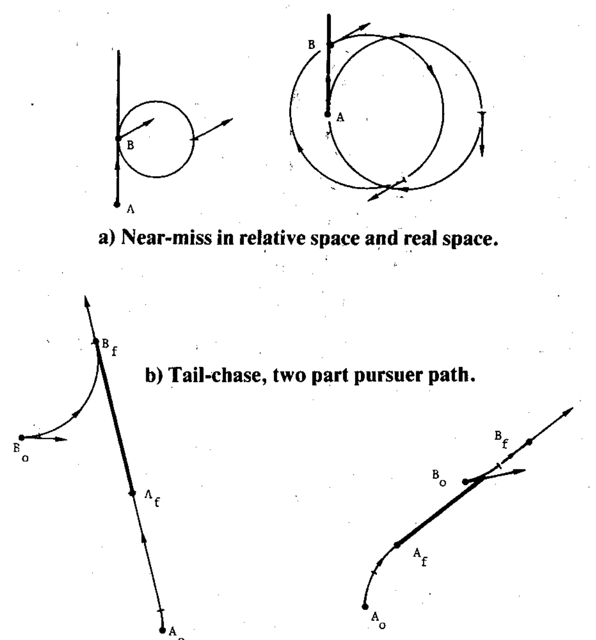


Fig. 2 Geometry of role specification at arbitrary heading.



a) Near-miss in relative space and real space.

b) Tail-chase, two part pursuer path.

c) Tail-chase, two-part evader path.

Fig. 3 Barrier maneuvers.

(x,y,H) using the Hamiltonian,

$$\mathcal{H} = \min_{\omega_A} \max_{\omega_B} (V_x\dot{x} + V_y\dot{y} + V_H\dot{H}) = 0 \tag{6}$$

The implication is that the controls must satisfy

$$\omega_A = \operatorname{sgn}(V_x y - V_y x + V_H)$$

$$\omega_B = \operatorname{sgn} V_H \tag{7}$$

where a positive turn rate for either A or B is to the right. In addition, the adjoint components must satisfy the differential equations

$$\dot{V}_x = -\omega_A V_y$$

$$\dot{V}_y = \omega_A V_x$$

$$\dot{V}_H = -V_x \cos H + V_y \sin H \tag{8}$$

where the terminal adjoints depend on the terminal geometry.

The "barrier" is a surface in the relative space composed of optimal trajectories which end with the evader just contacting the pursuer's kill zone in a tangential or "near-miss" manner. Optimal pursuit-evasion paths preceding these end states are *on* the barrier and are developed as the solution to the game of kind. This is followed by the solution to the game of degree, in which the final time is optimized by both players, for initial states *inside* the closed barrier. As mentioned earlier, the evader may also have the option of maneuvering for the simultaneous kill end condition from initial states inside the barrier. Important details of these solutions will be given for a specific value of weapon range, β.

The Game of Kind

As suggested by the preceding discussion, there exist four regions in the state space. Two of them are determined by near-miss barrier trajectories for which A pursues B and for which B pursues A. These paths intersect each other on the surfaces of Eqs. (2) and (3) which bisect the mutual-kill region. For such initial conditions, each must turn toward the other to avoid being killed prematurely. The remaining "draw" region outside the barrier is infinite because the speeds are equal by assumption.

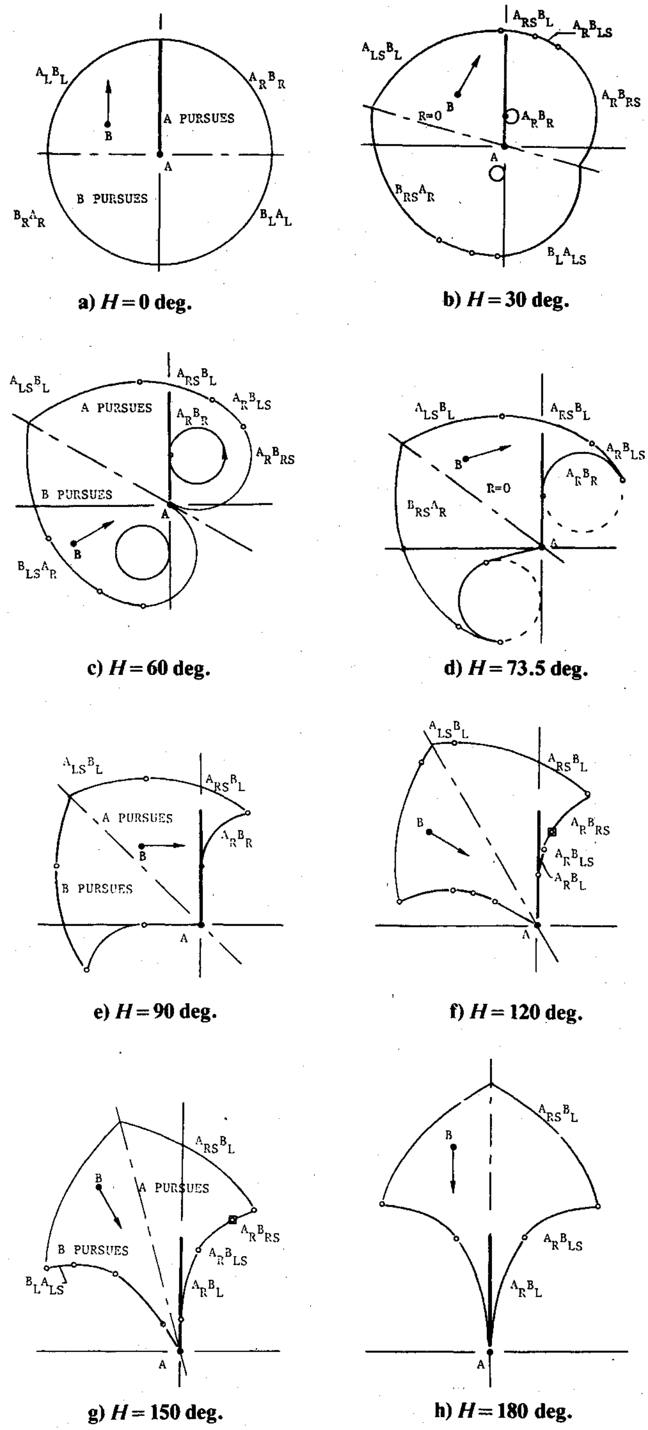
The solution to the game of kind, the barrier, is a rather complex two-dimensional surface in this third-order problem. It is a closed surface and those states inside the barrier are relative geometries for which the evader cannot avoid capture if the pursuer maneuvers optimally. Conversely, of course, the pursuer cannot bring the evader *inside* the closed barrier if the *evader* maneuvers optimally while in the draw region outside the barrier.

When A pursues, one type of near-miss trajectory ends with $x_f = \dot{x}_f = 0$, so that by Eq. (4), $y_f = \sin H_f$ when A is turning hard toward B, and B is turning hard away from A's weapon "envelope" or line segment of length β. The terminal adjoint vector is $(\pm 1, 0, 0)$ with the sign depending on whether B is to A's right or to A's left at this time. The switch functions of Eq. (7) do not change sign for this end condition, and relative-space and real-space paths are typically as shown in Fig. 3a.

A second type of barrier trajectory ends in the static tail-chase geometry with $y_f = \beta$ and $x_f = H_f = 0$, so the terminal adjoint vector is $(0,1,0)$.

Either A or B can precede this end condition with a dash, and a pair of real-space paths are shown in Figs. 3b and 3c for two sample cases when $\beta = 2$. Notice that evader B *cannot* use his weapon in these cases, because the conditions of Eqs. (2) and (3) are not met. If B were to do other than an optimal evasion, B would be killed at short range; this is the definition of the barrier.

Figure 4 shows cross sections of the three-dimensional barrier at eight relative headings in axes fixed to A and for $\beta = 2$. This value is chosen so that features of the solution will be clear, and not to represent a certain type of weapon. The notation gives pursuer and evader maneuvers, in that order; $A_{RS}B_L$, for example, means A pursues by following a right-straight path, while B turns left until termination at $(0,\beta,0)$. The role-reversal or mutual-kill surface $R=0$ is indicated by the dashed line at each heading. Increasing the range β increases the outer envelope size without changing the circular near-miss paths. Again, for the head-on geometry, Eq. (2), the roles are indeterminate for all relative positions. Hence the diagram of Fig. 4h actually corresponds to a relative heading just less than 180 deg.



a) $H=0$ deg. b) $H=30$ deg. c) $H=60$ deg. d) $H=73.5$ deg. e) $H=90$ deg. f) $H=120$ deg. g) $H=150$ deg. h) $H=180$ deg.

Fig. 4 Barrier cross sections; $\beta = 2$.

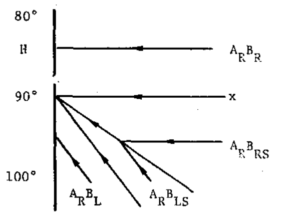
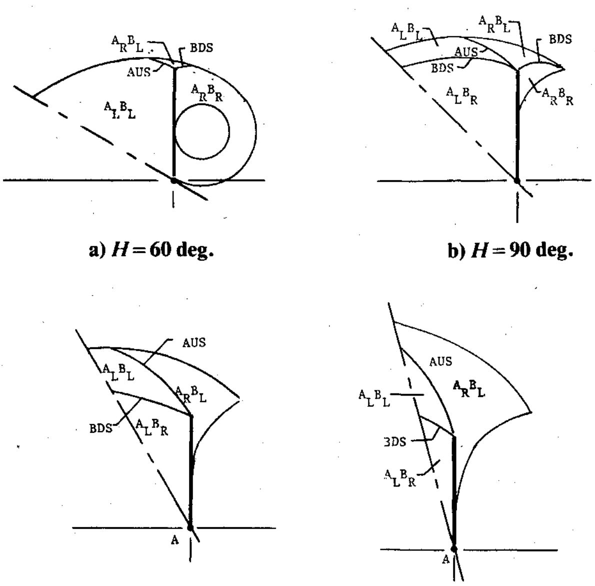


Fig. 5 Barrier trajectories near kill zone and $H=90$ deg.



a) $H=60$ deg. **b) $H=90$ deg.**

c) $H=120$ deg. **d) $H=150$ deg.**

Fig. 6 Optimal-time pursuit evasion maneuvers.

The near-miss barrier paths illustrated in Fig. 3a are found by using the solutions to the equations of motion for $\omega_A=\omega_B=1$. By eliminating the time to go from the solutions we find

$$(x-1+\cos H_f)^2+(y-\sin H_f)^2=(1-\cos H_f)^2 \tag{9}$$

Notice that H is constant when the turns are the same, so the relative motion stays at this heading. When inside these circles, the evader is in the "draw" region since the circles cease to be closed at the heading of Fig. 4d. These circles are tangent to A's kill zone, and apply only to relative headings below 90 deg, as shown in Figs. 4b-d.

The configuration of Fig. 4d is of special interest, since it shows the first occurrence of the evader's dispersal point on the barrier. At this heading, the concentric barrier segments of Figs. 4b and 4c coalesce to a single circle, and only the forward half remains as an actual barrier. When at the initial condition of the cusp, the evader must choose between a hard right turn ($\omega_B=1$) and a dash ($\omega_B=0$), while A takes $\omega_A=1$ for either choice. The resulting paths lead to end conditions $(0,\sin H^*,H^*)$ and $(0,\beta,0)$, respectively. By equating x and y coordinates as given by using these end conditions and controls in the retrograde equations of motion, we find the following analytic expression for the heading at which the cusp occurs:

$$H^*-\cos H^*=\beta-1 \tag{10}$$

For $\beta=2$, this yields $H^*=73.5$ deg as shown in Fig. 4d.

A second type of near-miss barrier trajectory results for more nearly "head-on" initial geometries, and for final headings between 90 and 180 deg. For the final heading $H_f=90$ deg, the necessary conditions show that the evader's *terminal* maneuver can be hard right, hard left, or dash ($\omega_B=1$, -1, or 0), while A turns right. But, for any heading $H>90$ deg, the barrier ends at $(0, \sin H_f, H_f)$, $H_f>90$ deg using optimal maneuvers A_RB_L. At this heading the barrier is extended for $H_f=90$ deg using two-part maneuvers A_RB_{LS} and A_RB_{RS}.

The barrier trajectories near $H=90$ deg are shown schematically in Fig. 5. These lines are relative motion paths which end at $x_f=\dot{x}_f=0$, for which only the (x,H) dependencies are indicated. The evader's singular arc and its tributaries are shown here more clearly. The coordinates of this singular-arc intersection with any plane $H>90$ deg are easily found as

$$x=1-\cos H+(H+1-\pi/2)\sin H$$

$$y=-\sin H-(H+1-\pi/2)\cos H \tag{11}$$

and this point is noted in Figs. 4f and 4g. For the "near" head-on geometry of Fig. 4h, this point is just outside the barrier $A_{RS}B_L$, and, therefore, does not occur on the barrier at this heading.

The remaining type of barrier end condition corresponds to the tail-chase geometry. Optimal controls preceding the end state $(0,\beta,0)$ require either pursuer or evader to use a turn-straight control sequence, depending on geometry. The dash maneuvers for either A or B are special cases occuring at smooth junctions of two segments of the barrier, as indicated in Figs. 4b and 4f. If maneuvers are *LS* and *RS* on either side of a junction, the optimal maneuver at the junction is a straight dash, *S*.

If the kill range is changed, the volume of the kill region must change in proportion; i.e., a larger β can never be a handicap to the pursuer. But the conical escape volume merely changes its length (i.e., the dispersal point angle H^*), since these circular sections do not involve the range β. The draw region to the pursuer's right for $H>90$ deg is also bounded by optimal paths independent of the weapon range, Eq. (11). But, at a given heading, the length of the adjacent A_RB_{RS} locus is an indirect function of weapon range. As shown for $H=150$ deg in Fig. 4, if the locus $A_{RS}B_L$ is modified by changing the weapon range, the dispersal-line intersection with the locus A_RB_{RS} also changes. Furthermore, a new feature arises when the weapon range is such that these surfaces no longer intersect. The capture region for $H>90$ deg then takes a larger but as yet unknown form.

The Game of Degree

When capture can occur, the pursuer and evader are presumably interested in optimizing the time to capture. An algorithm is sought for determining the controls for both which accomplish this objective. When A is the pursuer, the corresponding optimal controls are implied by the Hamiltonian, which is now the time derivative of the optimal capture time, which A is minimizing and B is maximizing,

$$\mathcal{H}=\min_{\omega_A}\max_{\omega_B}[V_x\dot{x}+V_y\dot{y}+V_H\dot{H}+1]=0 \tag{12}$$

The controls are then given, as in the game of kind, by

$$\omega_A=\operatorname{sgn}(V_xy-V_yx+V_H)$$

$$\omega_B=\operatorname{sgn}V_H \tag{13}$$

where again the adjoints satisfy Eq. (8) with final values to be specified by the terminal geometry.

There is a dash surface for A in the game of degree which ends with the evader at $(0,\beta,H_f)$. This singular arc control can be preceded by hard turns in either direction, and the corresponding surface is indicated in Fig. 6 by AUS for

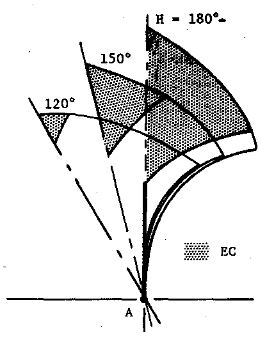


Fig. 7 Mutual-kill barrier at three headings.

"universal surface" in Isaacs' notation.[11] The maneuvers for the evader are left turns on and near this surface.

Two dispersal surfaces for the evader are also found inside the capture region. In both cases, B must choose between hard-left and hard-right turns when initially on this locus, while A's control is independent of B's choice. End conditions for these trajectories are $y_f = \beta$ and $y_f < \beta$, respectively. Both of these dispersal surfaces are shown at $H = 90$ deg in Fig. 6, although one exists only for $H < 90$ deg and the other one only for $H > 90$ deg.

The maneuvers of Fig. 6 are optimal in that the guidance law provides a saddle-point solution with respect to the capture time, with roles specified according to the function $R(x,y,H)$. But, there may be geometries[13] for which evader B can also choose to *pursue*, resulting in the mutual-kill end condition ($0,\beta,180$ deg). To find these states, a *second* barrier must be developed. It is found as the family of trajectories for which the "evader's" right turn is countered by a left-right sequence of the "pursuer," as A gets within range of the nominal evader just before the latter has completed his offensive turn. This barrier exists only for headings greater than about 100 deg, as shown in Fig. 7, and it bounds the region labeled EC, for "evader's choice." Any encounter beginning at locations inside this barrier results in A capturing B, but both tactics depend upon how the evader values his remaining maximum lifetime relative to an earlier mutual kill. Therefore, these short-range evader tactics can be subject to personal interpretation, and, in practice, some statistical considerations of the weapon systems would apply as well.

Summary and Conclusions

The purpose of this study was to determine a feasible method for role specification and combat maneuvers in the one-on-one aerial combat problem by the optimization principles of differential games. It was first noted that the relative position and relative velocity vectors for *any* geometry define a unique plane in which the principal control accelerations of both aircraft should be applied. The equations of relative motion, for coplanar motion at constant speeds, were then analyzed for the case in which the combat aircraft are *identical*. This led to identifying a role-determination function by geometric means alone. Use of this function, together with the common weapon range, then required real-time estimation by both aircraft of relative position and relative velocity vectors, in terms of which roles, optimal maneuvers, and capture regions for both were determined.

The capture regions, as computed by methods of differential games, of course depend in general on all of the problem parameters. For arbitrary nonsymmetric aircraft and weapons, even the third-order problem can require days of analysis of computer-generated trajectories. This is because only the *necessary* conditions are provided by the optimality criteria, and a large proportion of these candidate trajectories may not be used because portions of other trajectories are more suitable. The extreme complexity in the resulting solution is entirely inappropriate for application to the combat problem.

In this paper, results have been more simply obtained because many complexities are assumed away. For example, a sort of "average" pair of combat aircraft is studied. This, of course, means that neither can exploit the performance weaknesses of the other, since they are identical. The results are therefore graphically symmetrical. On the other hand, airborne sensors and computers may soon enable application of these results to actual aircraft/pilot engagements. This encourages the development of good combat maneuver algorithms.

The low-order aerial combat model has produced results which have a simple interpretation. First, the pursuit-evasion roles are specified according to the sign of a function of the range, bearing, and heading. The finite capture regions of both players are then found for these roles, and these regions are the states for which the nominal evader will be captured if the pursuer maneuvers as shown. Then, the controls for both players for states inside these regions are optimized with respect to the time to go. But from certain relative geometries, the evader can choose an earlier simultaneous kill strategy, and these regions are also found. The pursuer need be concerned by the evaders's weapon only for these geometries. The roles cannot be reversed unless the state passes *through* the mutual-kill surface which bisects the relative state space, and this can occur only if the pursuer is inattentive when the geometry is symmetric, on this surface.

At best, the analysis and results given here might be a first step in the actual implementation of combat guidance laws with a foundation in the min-max optimization theory of differential games. Modifications would be included, when necessary and possible, to account for departures from the present model. At worst, the results could be easily implemented in pilot ranging as the first *video* differential game. The name of the game would probably have to be Pursuit-Evasion Aerial Combat Maneuvers: PACMAN.

Acknowledgment

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